J. Fluid Mech. (1998), *vol.* 377, *pp.* 1–26. Printed in the United Kingdom

Analyses of slow high-concentration flows of granular materials

By S. B. SAVAGE

Department of Civil Engineering and Applied Mechanics, McGill University, Montreal H3A 2K6, Canada

(Received 15 February 1996 and in revised form 29 June 1998)

A theory intended for slow, dense flows of cohesionless granular materials is developed for the case of planar deformations. By considering granular flows on very fine scales, one can conveniently split the individual particle velocities into fluctuating and mean transport components, and employ the notion of granular temperature that plays a central role in rapid granular flows. On somewhat larger scales, one can think of analogous fluctuations in strain rates. Both kinds of fluctuations are utilized in the present paper. Following the standard continuum approach, the conservation equations for mass, momentum and particle translational fluctuation energy are presented. The latter two equations involve constitutive coefficients, whose determination is one of the main concerns of the present paper. We begin with an associated flow rule for the case of a compressible, frictional, plastic continuum. The functional dependence of the flow rule is chosen so that the limiting behaviours of the resulting constitutive relations are consistent with the results of the kinetic theories developed for rapid flow regimes. Following Hibler (1977) and assuming that there are fluctuations in the strain rates that have, for example, a Gaussian distribution function, it is possible to obtain a relationship between the mean stress and the mean strain rate. It turns out, perhaps surprisingly, that this relationship has a viscous-like character. For low shear rates, the constitutive behaviour is similar to that of a liquid in the sense that the effective viscosity decreases with increasing granular temperature, whereas for rapid granular flows, the viscosity increases with increasing granular temperature as in a gas. The rate of energy dissipation can be determined in a manner similar to that used to derive the viscosity coefficients. After assuming that the magnitude of the strain-rate fluctuations can be related to the granular temperature, we obtain a closed system of equations that can be used to solve boundary value problems. The theory is used to consider the case of a simple shear flow. The resulting expressions for the stress components are similar to models previously proposed on a more *ad hoc* basis in which quasi-static stress contributions were directly patched to rate-dependent stresses. The problem of slow granular flow in rough-walled vertical chutes is then considered and the velocity, concentration and granular temperature profiles are determined. Thin boundary layers next to the vertical sidewalls arise with the concentration boundary layer being thicker than the velocity boundary layer. This kind of behaviour is observed in both laboratory experiments and in granular dynamics simulations of vertical chute flows.

1. Introduction

1.1. *Kinetic theories for rapid granular flows*

Kinetic theories have been developed to treat rapid flows of granular materials for moderate and low concentrations (Savage & Jeffrey 1981; Jenkins & Savage 1983; Lun *et al.* 1984; Jenkins & Richman 1985; Jenkins 1987*a,b*; Savage 1995; Goldshtein & Shapiro 1995; Goldhirsch 1995; Goldshtein *et al.* 1995; Sela, Goldhirsch & Noscowicz 1996; Sela & Goldhirsch 1998). They are usually based on hard sphere models previously developed for dense fluids at the molecular level. We note that these treatments for dilute and dense gases, such as the Chapman–Enskog approximate solution of the Boltzmann equation, can be regarded as power series expansions in Knudsen number (cf. Woods 1993; Chapman & Cowling 1970). The kinetic energy associated with the translational velocity fluctuations of the granular particles has an obvious analogy with the definition of the temperature in a gas at the molecular level. We define the particle velocity fluctuation $\boldsymbol{c} = (\boldsymbol{v} - \boldsymbol{u})$, where $\boldsymbol{v}$ is the instantaneous particle velocity, $\boldsymbol{u} = \langle \boldsymbol{v} \rangle$ is the mean transport velocity and the angle brackets designate an ensemble average. The term *granular temperature* has been associated with the mean-square particle velocity fluctuations. We can define (for the three-dimensional case) a *translational* granular temperature $T = \langle c^2 \rangle /3$, such that $3T/2$ is the specific kinetic energy of the translational velocity fluctuations.

The particles involved in real granular flows are inelastic and have some finite roughness. Thus, when grains collide and rub against one another during flow, a significant amount of energy dissipation occurs. The need to account for this energy dissipation is the main difference between rapid granular flow and molecular modelling. The granular flow kinetic theories can be considered as perturbations of the case of perfectly elastic particles and the dissipation should be 'small' in some sense for the analyses to be valid. This corresponds to small values of the parameter R, where R is the product of a typical shear-rate, $(\mathrm{d}u/\mathrm{d}y)$, times the particle diameter d divided by the square root of the granular temperature; i.e. $R = (\mathrm{d}u/\mathrm{d}y)\,\mathrm{d}/T^{1/2} \ll 1$. In the case of the shear flow of a gas, R is very small and the expansion in powers of Knudsen number (or equivalently R) is quite legitimate. For granular flows, because of the relatively large dissipation, the granular temperature tends to decay unless it is maintained by shear work or an external source of energy such as vibration at the boundary walls. In some instances the values of R are not always small and can, in fact, be of order unity (Savage & Jeffrey 1981). As a result, some (for example, Clift 1993; Hinch 1995) have raised questions about the validity of granular flow kinetic theory approaches. However, numerous comparisons (see, for example, Walton & Braun 1986*a,b*; Walton, *et al.* 1987; Lun 1991; Savage 1992; Lun & Bent 1994; Lan & Rosato 1995) of the kinetic theory results with experimental measurements and granular dynamics computer simulations (which are free of the 'questionable' assumptions made in the kinetic theories) have shown quite good agreement for cases when the dissipation is relatively small and the particle concentrations range from small to moderate.

The above mentioned comparisons used the earlier kinetic theories that assumed the granular temperature to be isotropic. It is evident from the computer simulations that anisotropy occurs in the granular temperature as well as the normal stresses and that these anisotropies are particularly prominent at low concentrations. These observations led to the development of more general kinetic theories (Jenkins & Richman 1988; Richman 1989; Richman & Chou 1992; Goldhirsch & Sela 1996; Sela *et al.* 1996) to handle low densities and more dissipative particles. Through the introduction of an anisotropic Maxwellian velocity distribution function (Jenkins & Richman 1988; Richman 1989; Richman & Chou 1992), and by constructing the Chapman–Enskog expansion in a manner appropriate for granular materials and carrying it out to Burnett order (Goldhirsch & Sela 1996; Sela *et al.* 1996), these more elaborate kinetic theories were able predict the large normal stress differences observed at low concentrations. Hopkins & Shen (1992) have compared shear stresses, normal stresses and stress ratios predicted from the kinetic theory of Richman & Chou (1992) with their molecular dynamics and their Monte Carlo computer simulations for particles that range from nearly elastic (coefficient of restitution, $e = 0.9$) to very dissipative particles ($e = 0.3$). They rightly characterize the agreement between the three data sets as remarkable.

1.2. *Extensions to treat fluid–solid mixtures*

The single-phase granular flow kinetic theories have been extended to include the effects of interstitial fluids by adding extra terms to account for particle drag forces and energy dissipation arising from the differences between the fluid and particle velocities. The effective solids pressure and the effective solids viscosity can then be calculated in a systematic way. Sinclair & Jackson (1989) studied gas-particle flow in a vertical pipe and Louge, Mastorakos & Jenkins (1991) considered problems of pneumatic transport of particles in a turbulent fluid. A related two-phase flow analysis has been developed by Ding & Gidaspow (1990) (also, see Gidaspow 1994) to treat bubbling fluidized beds. The solids viscosities and stresses were obtained from the granular flow kinetic theory of Lun *et al.* (1984). These papers dealing with solid–fluid mixtures treated relatively massive particles in the presence of a gaseous fluid. Jenkins & McTigue (1990) and McTigue & Jenkins (1992) have given heuristic arguments to establish the form of the constitutive relations for concentrated suspensions of particles in liquids at low shear-rates corresponding to Bagnold's (1954) *macro–viscous* regime. Bagnold found that a distinctive feature of this flow regime was the existence of a dispersive normal stress that depended linearly upon shear-rate and was proportional to the shear stress. Jenkins & McTigue (1990) were able to find plausible and consistent functional forms for the transport coefficients by using dimensional and heuristic arguments. In a later analysis (Jenkins & McTigue 1995) they were able to determine more detailed, quantitative results by an approach analogous to those taken in the granular flow kinetic theories.

1.3. *Quasi-static stress contributions and particle interactions*

It is traditional to think of two limiting granular flow regimes: the fully dynamic *rapid flow* regime (called the *grain-inertia* regime by Bagnold 1954) and the *quasi-static*, rate-independent plastic regime (which has received extensive attention in the soil mechanics literature). Many practical flow situations, particularly those that are strongly affected by gravitational forces, fall into a transitional regime that lies between these two limiting flow regimes. It is extremely difficult to construct theoretical models of this flow regime and analyses performed to date have consisted of simple *ad hoc* patching together of results taken from the *grain-inertia* and the *quasi-static* regimes. Savage (1983) suggested that, for the case of a free-surface chute flow in a gravitational field, one might represent the total stresses as the linear sum of a rate-independent, dry friction part plus a rate-dependent 'viscous' part obtained from the high shear rate granular flow kinetic theories. The magnitude of the rate-independent contribution was chosen such that the sum of the two parts satisfied the overall momentum equation perpendicular to the flow direction. More detailed analyses based on this approach were presented by Johnson & Jackson (1987) and Johnson, Nott & Jackson (1990). While reasonable results were obtained for these particular problems, it is not obvious how the approach could be extended to handle more general kinds of flow problems.

The 'hard particle' assumption is a key element in the above mentioned rapid granular flow kinetic theories. In this limit of infinitely stiff particles, the collisional contact times tend to zero and thus only binary collisions need to be considered. The particle collision dynamics can be treated in a straightforward way and the evaluation of the Boltzmann collision integral and the determination of various transport coefficients are difficult but manageable. When the concentration is high and the deformation rates are low as in the quasi-static or transitional regimes, the particles typically experience multiple contacts that are long lasting rather than identifiable, short term, 'collisions'. One cannot analyse the particle interactions in a straightforward way as in the case of rapid flow. The particle interactions are not limited to binary ones. In experimental studies using photo-elastic disks and in two-dimensional computer simulations one can observe the development of force networks or chains involving large numbers of particles (Drescher & De Josselin De Jong 1972; Oda, Nemat-Nasser & Konishi 1985; Behringer & Baxter 1994; Gutfraind, Pouliquen & Savage 1995). Some particles are highly loaded and form chains, whereas others in between the chains are subjected to relatively small loads. When the bulk material deforms, particle contacts fade, new ones are generated, and the structure of the force networks has an apparently random transient character. While the force networks are clearly apparent in these two-dimensional disk-like systems, one should be cautious about inferring too much about the behaviour of more common, physically realistic, three-dimensional particles on the basis of these observations. In the three-dimensional case it is likely that the contact forces or stresses would be more homogeneous and that an isolated force perturbation or stress concentration would have a smaller domain of influence.

1.4. *Present investigation*

Because of (i) the complexity of the particle geometric configuration and its changing pattern, (ii) the long-range nature of the particle interactions, and (iii) the problems of handling the dynamics in such slow, closely packed flows, it is difficult to envisage how one might analyse these problems by other than stochastic approaches that are less detailed and less precise than, for example, the Chapman–Enskog procedure (Chapman & Cowling 1970). In an earlier version of the present paper, a preliminary attempt was made to formulate the simplest kind of analysis for slow, granular flows at low stress levels based on equations of the Fokker–Planck or Smoluchowski type (Eisenschitz 1958; Cole 1967; McQuarrie 1976). While it contained a number of *ad hoc* assumptions, it did yield the usual forms of the conservation equations for mass, momentum and translational velocity fluctuation energy in terms of stresses, energy fluxes rates of energy dissipation, etc. However, in contrast to the rapid granular flow kinetic theories, this stochastic approach could not provide the detailed forms of the constitutive coefficients, and it was necessary to determine them by separate analyses.

For reasons of brevity in the present paper, the conservation equations are presented following a simple, more standard continuum approach, but the essential issue is again that of determining the constitutive equations. Consideration is given to fluctuations in velocities at the individual particle level, as well as fluctuations in strain rate at a somewhat coarser scale. We approach the problem from the limit of low deformation rates and start with a compressible, frictional, plastic continuum that satisfies an associated flow rule. The functional dependence of the flow rule is chosen so that the

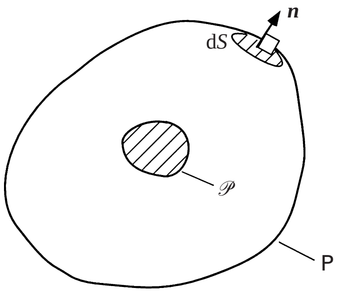

FIGURE 1. Diagram of the system P with surface element dS and surface normal $\boldsymbol{n}$, and the system element $\mathscr{P}$ associated with the volume element d$\boldsymbol{r}$.

limiting behaviours of the resulting constitutive relations are consistent with the results of the kinetic theories developed for rapid flow regimes. Following Hibler (1977), and assuming that there are fluctuations in the strain rates that have, for example, a Gaussian distribution function, it is possible to obtain a relationship between the mean stress and the mean strain rate, and hence to determine the viscosity coefficients. The rate of energy dissipation can be determined in a similar manner. With further assumptions that express the magnitude of the strain-rate fluctuations in terms of the granular temperature, the governing system of equations is closed. The resulting theory is first applied to consider a simple shear flow. The expressions derived for the stress components are compared to previous *ad hoc* models that directly patch together rate–independent and rate-dependent stress contributions. Next, the problem of slow granular flow in rough walled, vertical chutes is studied. The calculated velocity, concentration and granular temperature profiles exhibit thin, boundary layers adjacent to the vertical sidewalls. The predicted boundary layers have characteristics similar to those observed in laboratory experiments and in granular dynamics simulations.

2. Analysis

2.1. *Conservation equations*

For completeness we present a compact derivation of the governing conservation equations for mass, momentum and translational velocity fluctuation energy following the treatment of Woods (1975). Figure 1 shows the system P, a surface element dS and the outward normal to the surface $\boldsymbol{n}$. We suppose that the system element $\mathscr{P}$, associated with the volume element d$\boldsymbol{r}$, contains a sufficient number of particles so that 'continuum' quantities such as density, stresses, etc. can be defined. The instantaneous velocity of a particle $\boldsymbol{v}(\boldsymbol{r}, t)$ is decomposed into a slowly varying 'mean' part $\boldsymbol{u}(\boldsymbol{r}, t)$ and a rapidly fluctuating part $\boldsymbol{c}(\boldsymbol{r}, t)$,

$$\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{u}(\boldsymbol{r}, t) + \boldsymbol{c}(\boldsymbol{r}, t), \tag{2.1}$$

where $\boldsymbol{r}$ is the spatial coordinate and t is time. Taking the average of $\boldsymbol{v}(\boldsymbol{r}, t)$ yields the definition of $\boldsymbol{u}$

$$\boldsymbol{u} = \langle \boldsymbol{v}(\boldsymbol{r}, t) \rangle, \tag{2.2}$$

where the angle brackets designate an ensemble average. Note that we can express the total specific kinetic energy of the particles as

$$\frac{\langle \boldsymbol{v} \cdot \boldsymbol{v} \rangle}{2} = \frac{1}{2}(u^2 + \langle \boldsymbol{c} \cdot \boldsymbol{c} \rangle) = \frac{u^2}{2} + \frac{3T}{2}, \tag{2.3}$$

where $\langle \boldsymbol{c}\cdot\boldsymbol{c}\rangle/2 = 3T/2$ is the specific kinetic energy of the translational velocity fluctuations and T is the granular temperature.

To proceed with the development of the conservation equations, consider $\Phi(\boldsymbol{r},t)$ to be some property of a clump of the granular material. Then, we can write

$$\mathrm{d}\Phi = \left(\frac{\partial\Phi}{\partial t}\right)_{\boldsymbol{r}} \mathrm{d}t + \mathrm{d}\boldsymbol{r}\cdot\left(\frac{\partial\Phi}{\partial\boldsymbol{r}}\right)_{t}. \tag{2.4}$$

Noting that the 'average' flow velocity $\boldsymbol{u} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, we can express the material derivative at the convected point $P_c(\boldsymbol{r},t)$ as

$$\mathrm{D}\,\Phi = \left(\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)_{P_c} = \left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\Phi. \tag{2.5}$$

Now consider the convected derivative of the line element dx

$$\mathrm{D}(\mathrm{d}x) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}x\right)_{P_c} = \frac{\partial u_x}{\partial x}\mathrm{d}x. \tag{2.6}$$

Similarly, considering the line elements dy and dz yields

$$\mathrm{D}(\mathrm{d}y) = \frac{\partial u_y}{\partial y}\mathrm{d}y \quad \text{and} \quad \mathrm{D}(\mathrm{d}z) = \frac{\partial u_z}{\partial z}\mathrm{d}z, \tag{2.7}$$

and making use of these results for the convected derivatives, we obtain

$$\mathrm{D}(\mathrm{d}\boldsymbol{r}) = \mathrm{D}(\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z) = \mathrm{d}\boldsymbol{r}\nabla\cdot\boldsymbol{u}. \tag{2.8}$$

We can consider the material derivative of $\Phi(\boldsymbol{r},t) = \mathscr{X}(\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{r}$ that designates some physical property of the element $\mathscr{P}(\boldsymbol{r},t)$, and obtain the expression that is used to generate the conservation equations, i.e.

$$\begin{aligned}\frac{1}{\mathrm{d}\boldsymbol{r}}\mathrm{D}(\mathscr{X}\,\mathrm{d}\boldsymbol{r}) &= \mathrm{D}(\mathscr{X}) + \frac{\mathscr{X}}{\mathrm{d}\boldsymbol{r}}\mathrm{D}(\mathrm{d}\boldsymbol{r})\\ &= \mathrm{D}(\mathscr{X}) + \mathscr{X}\,\nabla\cdot\boldsymbol{u}.\end{aligned} \tag{2.9}$$

To obtain the *conservation of mass equation*, we take $\mathscr{X} = \rho$, the mass density of the bulk. Thus

$$\mathrm{D}(\rho\,\mathrm{d}\boldsymbol{r}) = \mathrm{d}\boldsymbol{r}\mathrm{D}(\rho) + \rho\,\mathrm{D}(\mathrm{d}\boldsymbol{r}) = 0, \tag{2.10}$$

and finally

$$\mathrm{D}(\rho) + \rho\,\nabla\cdot\boldsymbol{u} = \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\boldsymbol{u}) = 0. \tag{2.11}$$

Now let $\mathscr{X}$ be the momentum per unit volume $\rho\boldsymbol{u}$. It is changed by (*a*) the body forces per unit volume $\rho\,\boldsymbol{g}\,\mathrm{d}\boldsymbol{r}$, where $\boldsymbol{g}$ is the gravitational acceleration, and (*b*) the surface forces. The pressure tensor $\boldsymbol{\mathsf{p}}$ is defined such that $\boldsymbol{n}\cdot\boldsymbol{\mathsf{p}}\,\mathrm{d}S$ is the force transmitted across a surface element $\boldsymbol{n}\,\mathrm{d}S$ in the positive direction along $\boldsymbol{n}$. Hence, the surface integral of $-\boldsymbol{n}\cdot\boldsymbol{\mathsf{p}}\,\mathrm{d}S$ over the system P is the force acting on the material inside P by the material outside P. From the *Divergence Theorem* we infer that $-\nabla\cdot\boldsymbol{\mathsf{p}}\,\mathrm{d}\boldsymbol{r}$ is the corresponding force on the element $\mathscr{P}$. We thus obtain

$$\frac{1}{\mathrm{d}\boldsymbol{r}}\mathrm{D}(\rho\boldsymbol{u}\,\mathrm{d}\boldsymbol{r}) = \mathrm{D}(\rho\boldsymbol{u}) + \rho\boldsymbol{u}\nabla\cdot\boldsymbol{u} = \rho\boldsymbol{g} - \nabla\cdot\boldsymbol{\mathsf{p}}, \tag{2.12}$$

which after expanding and making use of the continuity equation (2.11) can be reduced to the standard form of the *linear momentum equation*

$$\rho\mathrm{D}(\boldsymbol{u}) = \rho\boldsymbol{g} - \nabla\cdot\boldsymbol{\mathsf{p}}. \tag{2.13}$$

It can be expressed in Cartesian tensor notation as

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=\rho g_i-\frac{\partial p_{ij}}{\partial x_j}. \tag{2.14}$$

Recall that the instantaneous velocity v_i can be expressed in terms of a mean part u_i and a fluctuating part c_i, such that $\langle \boldsymbol{v}\rangle=\boldsymbol{u}$ and the granular temperature $T=c^2/3$. Suppose we now take $\mathscr{X}$ to be the total kinetic energy per unit volume such that

$$\mathscr{X}=\rho\langle \boldsymbol{v}\cdot\boldsymbol{v}\rangle/2=\rho\left(u^2+3T\right)/2. \tag{2.15}$$

This kinetic energy is changed by: (*a*) work that the body force does on the element $\mathscr{P}$, (*b*) work and energy transfers to $\mathscr{P}$ by the contiguous particles, and (*c*) interparticle dissipation within $\mathscr{P}$. Expressing these contributions more explicitly we can write:

(*a*) The body force does work at the rate $\rho\boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{r}$ on the material within $\mathscr{P}$.

(*b*) (i) The flow of collisional fluctuation energy adds energy at the rate $-\nabla\cdot\boldsymbol{q}\,\mathrm{d}\boldsymbol{r}$, where $\boldsymbol{q}$ is the energy flux vector. (ii) The mechanical force that is transmitted across a convected surface element $\boldsymbol{n}\,\mathrm{d}S$ does work at a rate $-(\boldsymbol{n}\cdot\mathbf{p}\,\mathrm{d}S)\cdot\boldsymbol{u}$ on material on the negative side of the element. Hence the surface integral of $-\boldsymbol{n}\cdot\mathbf{p}\cdot\boldsymbol{u}\,\mathrm{d}S$ over the system is the rate at which the surface stress does work on the material inside P. By the divergence theorem, $-\nabla\cdot(\mathbf{p}\cdot\boldsymbol{u})\,\mathrm{d}\boldsymbol{r}$ is the rate that surface stresses do work on the element $\mathscr{P}$.

(*c*) Because of inelastic interparticle collisions and frictional rubbing, energy is dissipated at the rate $\gamma\,\mathrm{d}\boldsymbol{r}$, where γ is the rate of energy dissipation per unit volume.

Setting the rate of change of the total kinetic energy per unit volume to the sum of the above contributions we obtain

$$\frac{1}{\mathrm{d}\boldsymbol{r}}\mathrm{D}\left(\rho\left[u^2+3T\right]/2\right)=\rho\boldsymbol{g}\cdot\boldsymbol{u}-\nabla\cdot\boldsymbol{q}-\nabla\cdot(\mathbf{p}\cdot\boldsymbol{u})-\gamma. \tag{2.16}$$

By making use of the continuity (2.11) and linear momentum (2.13) equations, the *total translational energy equation* can be written in the following form

$$\rho\,\mathrm{D}\left(\frac{u^2}{2}+\frac{3T}{2}\right)=\rho\boldsymbol{g}\cdot\boldsymbol{u}-\nabla\cdot\boldsymbol{q}-\nabla\cdot(\mathbf{p}\cdot\boldsymbol{u})-\gamma. \tag{2.17}$$

Taking the vector product of $\boldsymbol{u}$ and the momentum equation (2.13) yields

$$\rho\,\mathrm{D}\left(\frac{u^2}{2}\right)=\rho\boldsymbol{g}\cdot\boldsymbol{u}-\boldsymbol{u}\cdot\nabla\cdot\mathbf{p}. \tag{2.18}$$

Subtracting this from (2.16), and making use of the vector relation

$$\nabla\cdot(\mathbf{p}\cdot\boldsymbol{u})-\boldsymbol{u}\cdot\nabla\cdot\mathbf{p}=\mathbf{p}^T:\nabla\boldsymbol{u}, \tag{2.19}$$

where $\mathbf{p}^T$ designates the transpose of $\mathbf{p}$, yields the *translational fluctuation energy equation*

$$\rho\,\mathrm{D}\left(\frac{3T}{2}\right)=-\mathbf{p}^T:\nabla\boldsymbol{u}-\nabla\cdot\boldsymbol{q}-\gamma. \tag{2.20}$$

Assuming the pressure tensor to be symmetric, $p_{ij}=p_{ji}$, we can express (2.20) in Cartesian tensor notation as

$$\frac{3}{2}\rho\left(\frac{\partial T}{\partial t}+u_j\frac{\partial T}{\partial x_j}\right)=-p_{ij}\frac{\partial u_i}{\partial x_j}-\frac{\partial q_j}{\partial x_j}-\gamma. \tag{2.21}$$

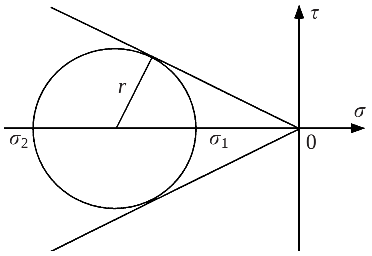

FIGURE 2. Mohr's circle of stress in (σ, τ)-space and yield lines for a cohesionless granular material with an internal friction angle ϕ.

2.2. *Yield function and constitutive relations*

The linear momentum and the translational fluctuation energy equations just presented contain the pressure tensor $\boldsymbol{p}$, the energy flux vector $\boldsymbol{q}$, and the rate of energy dissipation per unit volume γ. Explicit constitutive equations for $\boldsymbol{p}$, $\boldsymbol{q}$, and γ and the associated constitutive coefficients are now derived. The analysis begins with an associated flow rule similar in form to what is typically used to model a compressible, frictional, plastic continuum undergoing quasi-static deformations. By assuming that there are fluctuations in strain rate, and then averaging, we can obtain a relationship between mean stress and the mean strain rate. It is common to think of rate-independence in connection with perfectly plastic, slow deformation of granular materials. However, we show, following Hibler (1977), that by introducing strain-rate fluctuations into the constitutive equations and taking statistical averages we can obtain a rate-dependent, viscous-like behaviour even for relatively small deformation rates. Some care is taken in formulating the functional dependence of the flow rule so that the behaviour at high shear rates is consistent with the results of the kinetic theories for granular flows.

In formulating the yield function, we shall make use of the ideas of critical state soil mechanics (Schofield & Wroth 1968; Chen & Mizuno 1990; Wood 1990; also see Lade & Prabucki 1995, for more recent references) and the work of Hibler (1977, 1979). For simplicity, attention is restricted in the present paper to the case of *two-dimensional* flows. In this planar case, we define the mean stress p, and the stress difference or maximum shear stress q as

$$p = \frac{\sigma_1 + \sigma_2}{2}, \quad q = \frac{\sigma_1 - \sigma_2}{2}, \tag{2.22}$$

where σ_1 and σ_2 are the major and minor principal stresses respectively (with tensile stresses taken as positive). Figure 2 illustrates Mohr's circle of stress in (normal stress σ, shear stress τ)-space and the yield lines for a cohesionless granular material with an internal friction angle of ϕ. The radius of the Mohr's circle is given by $r = q = (\sigma_1 - \sigma_2)/2$ and $r = -p \sin\phi$.

Figure 3 shows an elliptical yield function in (p, q)-space corresponding to a particular set of values of solids fraction ν, and granular temperature T. As will be discussed subsequently, the size of the yield envelope grows with increasing solids fraction and granular temperature. In the present work, an elliptical yield envelope is chosen for reasons of simplicity; but, we note that other shapes, such as teardrops and parabolas, are also commonly used (Wood 1990; Lade & Prabucki 1995; Bolzon, Schrefler & Zienkiewicz 1996). Elliptical yield surfaces often have been used in the past for soil mechanics problems (Schofield & Wroth 1968; Mroz, Norris &

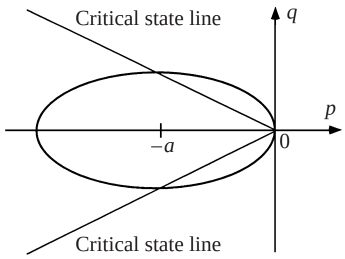

FIGURE 3. Elliptical yield function in (p, q)-space for a particular set of values of solids fraction ν, and granular temperature T. An infinite set of nested yield functions exists, corresponding to other values of solids fraction and granular temperature.

Zienkiewicz 1979; Chen & Mizuno 1990; Wood 1990). The commercial finite element package ABAQUS (1994) uses elliptical yield functions to model pressure-dependent yielding in materials including soils and foams. Hibler's (1979) elliptical yield function is the standard approach used to model the internal stresses developed in pack ice floating on arctic waters.

The critical state soil mechanics models were formulated in part to predict volume changes associated with different kinds of loading histories (see Wood 1990 for a lucid discussion of these models). One thinks of families of yield surfaces and plastic potential curves that are expressed in terms of parameters that control the sizes of particular members of these surfaces. By making the assumption of an *associated flow rule*, in which the plastic potential is equal to (i.e. 'associated' with) the yield function, the 'plastic' strains or strain rates can be derived from a plastic potential function by differentiating the plastic potential with respect to the stress components. More specifically, for *associated flow* the material obeys the postulate of *normality* in which the plastic strain increment is in the direction of the outward normal to the yield surface. To clarify this, let us consider a point on the yield locus shown in figure 3 and examine the projection of the outward normals to the yield locus on the p-axis. If this projection is negative (outward away from the origin) then compaction will occur during deformation, i.e. ν will increase. If the projection is positive (toward the origin), then dilation will occur during deformation and ν decreases. At the so-called *critical state*, the normal is perpendicular to the p-axis and deformation occurs without change in volume. Along the *critical state* line (cf. Figure 3) we write $|q| = -p \sin \phi$, where ϕ is the *critical state* internal friction angle. This expression has the same form as the Mohr–Coulomb relation for an incompressible, cohesionless granular material. However, note that the angle between the critical state line and the p-axis in figure 3 is given by arctan(sin ϕ).

It should be noted that while the critical state models and the associated flow rule assumption are extremely useful and illuminating representations of soil behaviour, they do have their limitations. In the initial, quasi-static yielding of sands involving relative small strains, the assumption of normality is less appropriate and constitutive models involving distinctly different shapes for the yield loci and plastic potentials have been developed (cf. Lade 1977; Vermeer 1984; Lade & Pradel 1990; Pradel & Lade 1990). The principal axes of stress and strain do not necessarily coincide in these kinds of models as they do in the case of the associated flow models.

The focus of the present paper is on continued flows that involve very large strains as well as deformation rates that are considerably larger than those typical in quasi-static soil mechanics applications. The aim is to develop a simple model to handle transitional flows that span the quasi-static and collisional flow regimes. This is difficult because there is virtually no detailed experimental information to guide the formulation of constitutive models to handle these flows. While the present model may not be sufficiently general to accurately predict *quasi-static* ‘localization’ phenomena such as shear band formation, that is not its purpose. We merely seek a simple representation of the behaviour for slow, concentrated granular flows that has a form that can be merged smoothly with the kinetic theory results for rapid, collisional flows. The main interest of the present paper is in predicting transitional and rapid flows.

Note that in the kinetic theories for rapid granular flows, the principal axes of stress and strain rate coincide. As will be seen, the introduction of strain rate fluctuations into the simple critical state model based on associated flow yields expressions for stresses that resemble those for a compressible, Newtonian fluid. It is possible to combine these expressions with the rapid flow kinetic theory results in a straightforward manner.

To proceed, we now define an elliptical yield function in terms of p and q as

$$F(p,q) = (p+a)^2 + e^2q^2 - a^2 = 0, \tag{2.23}$$

where e is the ellipticity or the yield curve major to minor axis ratio, and $a = a(v, T)$ is the value of p at the centre of the ellipse. On the critical state line shown in figure 3, $p = -a$, and $e^2q^2 = a^2$; hence

$$\frac{|q|}{a} = \frac{1}{e} = \sin\phi, \tag{2.24}$$

where ϕ corresponds to the value of the internal friction angle on critical state line.

Also, we can express the yield function (2.23) in terms of the stress components σ_{ij} referred to an (x, y) coordinate system as

$$F(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = \left[\frac{\sigma_{xx}+\sigma_{yy}}{2} + a\right]^2 + e^2\left[\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2\right] - a^2 = 0. \tag{2.25}$$

Now define the strain rates as

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]. \tag{2.26}$$

In the usual way, the ‘plastic’ strain rates can be derived from a plastic potential function and we make the assumption of an associated flow rule, in which the plastic potential is equal to the yield function. Thus, the plastic strain rates are given by

$$e_{ij} = \lambda\frac{\partial F}{\partial \sigma_{ij}}. \tag{2.27}$$

For example

$$e_{xx} = \lambda\left[2\left(\frac{\sigma_{xx}+\sigma_{yy}}{2} + a\right) + \frac{e^2}{2}(\sigma_{xx}-\sigma_{yy})\right], \tag{2.28}$$

and we can write down similar expressions for e_{yy}, e_{xy}, e_{yx}. Making use of these expressions for strain rates, solving for stress components in terms of strain rates, and then substituting in the yield function (2.25) we find

$$\sigma_{ij} = -\frac{a}{\Delta}\left[\frac{2}{e^2}e_{ij} + \delta_{ij}\left(1-\frac{1}{e^2}\right)e_{kk}\right] - \delta_{ij}a, \tag{2.29}$$

where

$$\Delta = \left[(e_{xx}^2 + e_{yy}^2) \left(1 + \frac{1}{e^2} \right) + \frac{4e_{xy}}{e^2} + 2e_{xx}e_{yy} \left(1 - \frac{1}{e^2} \right) \right]^{1/2}. \tag{2.30}$$

Note that the principal axes of stress and strain rate coincide. As expressed above, it may be observed that the stresses have a rate-independent form. However, a viscous-like behaviour can be produced by the procedure described below.

Following Hibler (1977), we assume that there exist fluctuations in the strain rate e_{ij} about the mean value $\langle e_{ij} \rangle$. The strain rate fluctuations are analogous to the particle velocity fluctuations $\boldsymbol{c}$ about their mean velocity $\boldsymbol{u}$. We assume that the strain rate distribution function $f(e_{ij})$ for the fluctuations is Gaussian (analogous to the Maxwellian velocity distribution function); hence

$$f(e_{ij}) = \frac{1}{\epsilon(2\pi)^{1/2}} \exp\left[-\frac{\left(e_{ij} - \langle e_{ij} \rangle\right)^2}{2\epsilon^2} \right], \tag{2.31}$$

where ϵ is the standard deviation of the strain rate fluctuations that have been assumed to be isotropic. Note that ϵ^2 is analogous to the granular temperature T, and that $e_{ij} = e_{ji}$.

The mean value of stress can be obtained by multiplying σ_{ij} by $f(e_{ij})$ and integrating over strain rate space. Hence

$$\langle \sigma_{ij} \rangle = \frac{1}{\epsilon^3(2\pi)^{3/2}} \int \exp\left(-\frac{\left(e_{xx} - \langle e_{xx} \rangle\right)^2 + \left(e_{yy} - \langle e_{yy} \rangle\right)^2 + \left(e_{xy} - \langle e_{xy} \rangle\right)^2}{2\epsilon^2} \right) \times \sigma_{ij} \, \mathrm{d}e_{xx} \, \mathrm{d}e_{yy} \, \mathrm{d}e_{xy}. \tag{2.32}$$

As Hibler (1977) has noted, we can imagine two limiting cases. First, considering (2.32) for very small strain-rate fluctuations ϵ, we see that we can replace e_{ij} by $\langle e_{ij} \rangle$ in the expression (2.29) for σ_{ij}, and thus we obtain a 'rate-independent', plastic behaviour. For values of ϵ that are large compared to the mean strain rates $\langle e_{ij} \rangle$, we can expand the right-hand side of (2.32) in powers of $\langle e_{ij} \rangle / \epsilon$. This yields, to first order, a linear viscous behaviour. We can evaluate the first-order terms in a reasonably straightforward way following Hibler (1977).

For example, consider the expression for $\langle \sigma_{xx} \rangle$ from (2.32) which is a function of $\langle e_{xx} \rangle$, $\langle e_{yy} \rangle$, and $\langle e_{xy} \rangle$. Note that from symmetry

$$\left. \frac{\partial \langle \sigma_{xx} \rangle}{\partial e_{xy}} \right|_{\langle \mathbf{e} \rangle \to 0} = \left. \frac{\partial \langle \sigma_{xx} \rangle}{\partial e_{yy}} \right|_{\langle \mathbf{e} \rangle \to 0} = 0, \tag{2.33}$$

where the subscript $\langle \mathbf{e} \rangle$ is the mean strain-rate tensor. Hence, to determine $\langle \sigma_{xx} \rangle$ to first order in strain rates, we need only consider the expression

$$\left. \frac{\partial \langle \sigma_{xx} \rangle}{\partial e_{xx}} \right|_{\langle \mathbf{e} \rangle \to 0} \tag{2.34}$$

and evaluate the resulting integral over e_{xx}, e_{yy}, e_{xy}. For the case in which the eccentricity $e = 1$, we can obtain a closed form solution; otherwise we must integrate numerically. It is convenient to transform to spherical coordinates by introducing

$$e_{xx} = \epsilon \, r \cos\psi \sin\theta, \quad e_{yy} = \epsilon \, r \sin\psi \sin\theta, \quad e_{xy} = \epsilon \, r \cos\theta, \tag{2.35}$$

and integrate over $0 < r < \infty$, $0 < \theta < \pi$, $0 < \psi < 2\pi$.

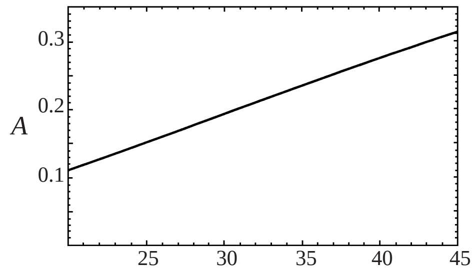

FIGURE 4. Variation of coefficient A with the critical state friction angle ϕ.

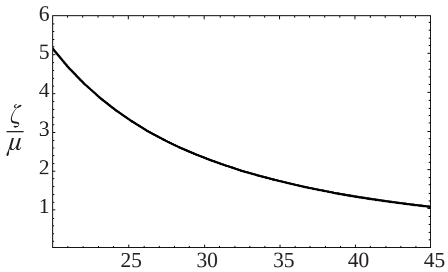

FIGURE 5. Variation of the ratio of viscosity coefficients ζ/μ with the critical state friction angle ϕ.

By proceeding in a similar way to determine $\langle\sigma_{yy}\rangle$ and $\langle\sigma_{xy}\rangle$ we can write the mean stress tensor to first order in $\langle e_{ij}\rangle$ as

$$\langle\sigma_{ij}\rangle = 2\mu\langle e_{ij}\rangle + (\zeta-\mu)\langle e_{kk}\rangle\delta_{ij} - a\delta_{ij}. \tag{2.36}$$

This has the same form as the stress–strain-rate relation for a Newtonian viscous fluid where μ is the shear viscosity and $(\zeta-\mu)$ is (sometimes) called the bulk viscosity. These viscosity coefficients have the forms

$$\mu = \frac{aA}{\epsilon}, \quad \zeta = \frac{aB}{\epsilon}, \tag{2.37}$$

where A and B depend only on the critical state friction angle ϕ and are determined by numerical integration. Figures 4 and 5 respectively show the variations of the coefficient A in the expression for shear viscosity and the ratio of ζ/μ with the critical state friction angle ϕ.

We can also calculate energy dissipation due to strain-rate fluctuations in a similar fashion. Write

$$\sigma_{ij} = \langle\sigma_{ij}\rangle + \sigma'_{ij}, \quad e_{ij} = \langle e_{ij}\rangle + e'_{ij}, \tag{2.38}$$

where the primes designate the fluctuating parts. Thus, the rate of energy dissipation γ due to fluctuations can be written as

$$\gamma = \langle\sigma'_{ij}e'_{ij}\rangle = a\epsilon D, \tag{2.39}$$

where the coefficient D is determined by numerical integration and is a function of the critical state friction angle ϕ (cf. figure 6).

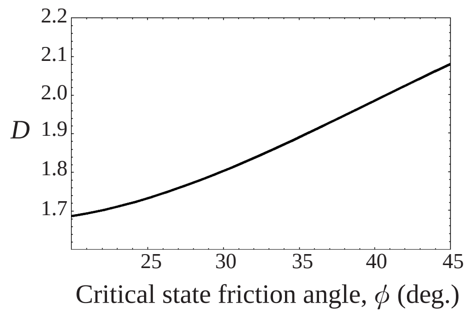

FIGURE 6. Variation of coefficient D appearing in the expression for dissipation with the critical state friction angle ϕ.

An explicit form for $a(\nu, T)$ is required. The yield ellipse grows in size with an increase in solids fraction and also with an increase in granular temperature (which corresponds to an increase in pressure). We assume that $a(\nu, T)$ is composed of the sum of these two contributions as follows:

$$a(\nu, T) = a_\nu(\nu) + a_T(\nu, T). \tag{2.40}$$

Schofield & Wroth (1968) suggested a pressure density relation for quasi-static deformations of the form

$$\frac{1}{\rho} = A - \log p. \tag{2.41}$$

However, this is not so appropriate for low stress levels. Jenike (1961, 1987) proposed an expression more suitable for low stresses of the form

$$\rho = C(p - p_0)^\beta, \quad \text{where} \quad \beta \simeq 0.05\text{–}0.1. \tag{2.42}$$

Related expressions have been proposed by Cowin (1977), Sundaram & Cowin (1979) and Kruyt (1990). We shall use a similar expression here:

$$a_\nu(\nu) = a_0 \log\left[\frac{\nu_\infty - \nu_0}{\nu_\infty - \nu}\right], \tag{2.43}$$

where a_0 is a reference value of a, ν_∞ is the solids fraction corresponding to closest packing, and ν_0 is the minimum solids fraction. We add to this a collisional stress contribution that depends on granular temperature of the form

$$a_T(\nu, T) = \rho_s \nu(1 + 2G)T, \tag{2.44}$$

where $\rho = \rho_s \nu$, ρ_s is the mass density of individual particles. The collisional stress contribution (2.44) is taken from the expression for pressure from the granular flow kinetic theory of Jenkins (1987*a*) for disks. Note that for the flow of disks, we define $T = \langle c^2\rangle/2$. In Jenkins (1987*a*) the expression for G is given as

$$G = \nu g(\nu) = \frac{\nu(16 - 7\nu)}{16(1 - \nu)^2}, \tag{2.45}$$

where $g(\nu)$ is the radial distribution function at contact. In the present work we modify this slightly and write

$$G = \nu g(\nu) = \frac{\nu(16 - 7\nu)}{16(1 - \nu/\nu_\infty)^2}, \tag{2.46}$$

so that the stresses diverge as the solids fraction approaches the maximum value. This form of expression for the radial distribution has been used several times in the past (for example, see Lun & Savage 1987; Lun 1991; Anderson & Jackson 1992; Gidaspow 1994). It makes the collisional contribution (2.44) consistent with the quasi-static contribution (2.43) in the sense that both diverge as $\nu \to \nu_\infty$.

For high deformation rates and large granular temperatures, this collisional term is dominant and the stresses will tend to the kinetic theory results. Thus, adding the two contributions (2.43) and (2.44) we obtain

$$a(\nu, T) = a_0 \log \left[\frac{\nu_\infty - \nu_0}{\nu_\infty - \nu} \right] + \rho_s \nu (1 + 2G) T. \tag{2.47}$$

We shall now relate the root-mean-square strain-rate fluctuations ϵ to the granular temperature T. If we were considering, for example, a gas where there is a wide separation between length scales associated with particle fluctuations and the length scales associated with fluctuations in strain rate, there would not necessarily be any connection between ϵ and T. However, for granular flows there is not such a wide disparity between the microscale, i.e. the particle diameter d, and the macroscale, which typically might be associated with the thickness of a shear layer. Usually shear layers are found to be of the order of 10 particle diameters in thickness and here it is plausible to think of a more direct link between ϵ and T. Notice that the dimensions of ϵ are the same as (velocity/length), i.e. the same as $T^{1/2}/d$. Here we shall assume the simple relationship

$$\epsilon = \frac{\beta T^{1/2}}{d}, \tag{2.48}$$

where β is a constant of order unity and d is particle diameter. Using (2.48) we can write the expression for shear viscosity μ in (2.37) as

$$\mu = \frac{aAd}{\beta T^{1/2}}. \tag{2.49}$$

We choose a value of β such that, as T becomes large, the above expression for shear viscosity μ agrees with the collisional kinetic theory result for disks. The viscosity from the kinetic theory of Jenkins (1987*a*) is

$$\mu = \left[1 + \frac{\pi}{8} (G^{-2} + 2G^{-1} + 1) \right] \frac{\rho_s \, \mathrm{d} \, \nu G T^{1/2}}{\pi^{1/2}}. \tag{2.50}$$

For large T, the expression (2.47) for $a(\nu, T)$ takes the form

$$a(\nu, T) \to \rho_s \nu (1 + 2G) T. \tag{2.51}$$

Using this result and equating the two expressions (2.49) and (2.50) for μ we obtain

$$\beta = \frac{(1 + 2G) A \pi^{1/2}}{G \left[1 + \frac{1}{8} \pi (G^{-2} + 2G^{-1} + 1) \right]}. \tag{2.52}$$

Conveniently, it is found that $\beta \simeq 1/4$ for solids fractions $0.65 < \nu < 0.9$.

Also, from Jenkins' (1987) kinetic theory for disks, the ratio of conductivity k to shear viscosity μ is

$$\frac{k}{\mu} = \frac{2 \left[1 + \frac{1}{4} \pi (G^{-2} + 3G^{-1} + 9/4) \right]}{\left[1 + \frac{1}{8} \pi (G^{-2} + 2G^{-1} + 1) \right]}, \tag{2.53}$$

which similarly has a nearly constant value $k/\mu \simeq 4$ for solids fractions $0.65 < \nu < 0.9$. We shall use the same value of this ratio for the present analysis.

Finally, substituting (2.48) in (2.39), the rate of energy dissipation becomes

$$\gamma = \frac{a\beta T^{1/2} D}{d}. \tag{2.54}$$

3. Simple shear flow

As a very simple example, we now apply the equations just derived in §2 to consider the case of simple shear flow ($\mathrm{d}u/\mathrm{d}y = \text{const.}$) in which the granular temperature T and solids fraction ν are both constant.

After making use of (2.36), the translational fluctuation energy equation (2.21) reduces to

$$\mu \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 - \gamma = 0. \tag{3.1}$$

Substituting the expressions (2.49) and (2.54) for μ and γ into (3.1) we obtain

$$T^{1/2} = \frac{d}{\beta}\left(\frac{A}{D}\right)^{1/2}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right) = \mathrm{d}K^{1/2}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right), \tag{3.2}$$

where $K = (A/D)/\beta^2$. Now making use of (2.49), (3.2) and the expression (2.47) for $a(\nu, T)$ we can rewrite the shear stress from (2.36) as

$$\begin{aligned}\sigma_{xy} &= \mu\frac{\mathrm{d}u}{\mathrm{d}y} = a(AD)^{1/2} \\ &= (AD)^{1/2}\left[a_0 \log\left[\frac{\nu_\infty - \nu_0}{\nu_\infty - \nu}\right] + \rho_s \nu(1+2G)d^2 K\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\right].\end{aligned} \tag{3.3}$$

This has the form of an extended Bingham-type fluid with a power law dependence on shear rate that has previously been proposed to describe the behaviour of granular materials (Savage 1979; Takahashi 1981, 1991; Tsubaki & Hashimoto 1983; Chen 1988). For a particular solids fraction ν, the first term on the right-hand side of (3.3) is a constant term that is independent of shear rate and the second term depends on the square of the shear rate as in the classical paper of Bagnold (1954). The normal stresses in the x- and y-directions are equal and given by

$$\sigma_{xx} = \sigma_{yy} = a, \tag{3.4}$$

in which a is composed of a rate-independent term and a term that depends on the square of the shear rate. The ratio of shear to normal stress is a constant given by

$$\frac{\sigma_{xy}}{\sigma_{yy}} = (AD)^{1/2}. \tag{3.5}$$

Figure 7 compares the ratio of shear to normal stress given by (3.5) for critical state internal friction angles $\phi = 20°$ and $25°$ with the result given by the granular flow kinetic theory of Jenkins (1987*a*) for values of the coefficient of restitution $e_r = 0.7$ and 0.8. In the rapid flow regime, e_r is a parameter that characterizes the energy dissipation, whereas the critical state internal friction angles ϕ plays that role in the present analysis. The important thing to note is that the ratios of shear to normal stress given by these two analyses are quite similar.

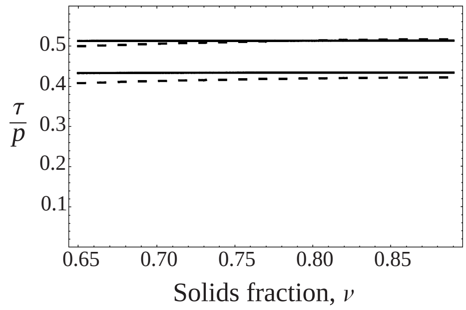

FIGURE 7. Ratio of shear to normal stress versus solids fraction. Upper and lower solid lines correspond to equation (3.5) for critical state internal friction angles ϕ of 25° and 20° respectively. Upper and lower dashed lines correspond to the granular flow kinetic theory of Jenkins (1987*a*) for values of the coefficient of restitution e_r of 0.7 and 0.8 respectively.

4. Vertical channel flow

We now consider slow, steady granular flow down a vertical channel having rough parallel sidewalls. This is a deceptively simple problem, but a particularly interesting example because in the slow plastic flow regime it has certain paradoxical aspects (Savage 1995). Experiments (cf. Takahashi & Yanai 1973; Nedderman & Laohakul 1980; Savage 1979, 1984; Natarajan, Hunt & Taylor 1995; Pouliquen & Gutfraind 1996) have found that the velocity profiles consist of two relatively thin boundary layers (typically between 5 and 15 particle diameters thick) next to the rough sidewalls, and a plug-like, uniform-velocity central region. When the gap between the vertical sidewalls is increased, the width of the central plug flow region is correspondingly increased while the shear layer thicknesses remain almost constant. Deep down in the bin where there is no longer any variation of the stresses in the vertical direction, the equilibrium equations show that for a uniform density of the material, the shear stress varies linearly across the width of the channel and the normal stress in the horizontal direction is constant. For a steady, fully developed flow, having the observed velocity profiles, it is natural to imagine that the shear planes are parallel to the vertical sidewalls. For a Mohr–Coulomb material yielding in such a way, the ratio of shear to normal stress on the shear planes would have to be constant. However, this is in conflict with the variations of shear and normal stresses required to satisfy the equilibrium equations. The only solution that is realizable is one in which the plug flow region spans the entire width of the channel, the boundary layers have zero thickness, and there is slip at the sidewalls. All of the interior material behaves as a rigid material; the stress states are inside the yield envelope.

For viscous fluid flow, of course, no such dilemma occurs. For rapid granular flows, which have certain viscous-like characteristics and more importantly can generate 'dispersive' normal stresses, it is also possible to find solutions for stress and velocity distributions. Hui & Haff (1986) have analysed vertical channel flow using Haff's (1983) heuristic kinetic grain flow theory and found plausible results, although there are some inconsistencies for the case of dissipative particles when the granular temperature (which is responsible for the generation of the normal stresses) goes to zero in the central region in wide channels. Savage (1995) has suggested that consideration of stress and velocity fluctuations for the case of slow, quasistatic flows may be necessary to analyse these flows correctly. Pouliquen & Gutfraind (1996) have attempted to treat the vertical channel problem using an adaptation of Eyring's

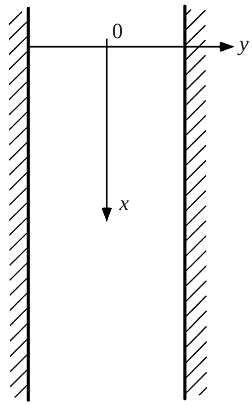

FIGURE 8. Definition sketch for rough walled vertical bin of half-width b.

(Krausz & Eyring 1975) rate theory originally developed to describe the viscosity of liquids. However the leap from Eyring's theory which involves vibrating particles in a lattice overcoming energy barriers to the consideration of deformation of irregularly packed, dissipative granular material seems tenuous. Finally, it is noted that models of structured continua, such as a Cosserat material (cf. Mühlhaus & Vardoulakis 1987; Vardoulakis & Sulem 1995) and strain-gradient theories (Fleck & Hutchinson 1997) also predict shear bands of finite thickness, but the writer is not aware of any analyses of the vertical chute problem using such models.

4.1. *Governing equations for vertical chute flow*

The theory developed in §2 is now applied to treat the flow down a vertical channel. Consider the x-coordinate to be vertical, increasing downward, and y to be directed horizontally as shown in figure 8. For steady, fully developed, two-dimensional flow

$$\partial/\partial t = \partial/\partial x = 0, \tag{4.1}$$

and it follows that

$$u = u(y), \quad v = 0, \quad T = T(y), \quad \nu = \nu(y). \tag{4.2}$$

The continuity equation (2.11)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{4.3}$$

is identically satisfied. The x-component of the linear momentum equation (2.13) after making use of (2.36) is

$$\rho \frac{\mathrm{d}u}{\mathrm{d}t} = 0 = \rho g + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = \rho g + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \tag{4.4}$$

and the y-component is

$$\rho \frac{\mathrm{d}v}{\mathrm{d}t} = 0 = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \frac{\partial \sigma_{yy}}{\partial y}. \tag{4.5}$$

After making use of (2.36), the fluctuation energy equation (2.21) can be written as

$$\rho \frac{\mathrm{d}T}{\mathrm{d}t} = 0 = \sigma_{xy}\frac{\partial u}{\partial y} - \frac{\partial q_y}{\partial y} - \gamma = \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) - \gamma. \tag{4.6}$$

We can integrate the x-momentum equation (4.4) to obtain

$$\sigma_{xy} = \mu \frac{\partial u}{\partial y} = -\int_0^y \rho g \, \mathrm{d}y. \tag{4.7}$$

Integrating the y-momentum equation (4.5) we obtain

$$\sigma_{yy} = \text{const.} = -a(v, T) = \sigma_{xx}. \tag{4.8}$$

Note that the molecular dynamics computer simulations of Gutfraind & Pouliquen (1996) found that the averaged normal stresses σ_{xx} and σ_{yy} were equal (within the resolution of the numerical results) and approximately constant over the width of the bin, as predicted by (4.8).

4.2. *Boundary conditions*

To proceed further with the analysis, we must take some account of the boundary conditions at the centreline and at the sidewalls. The conditions at the centreline are evident either from symmetry considerations (derivatives of velocity, solids fraction and granular temperature with respect to y are zero), or are free to be chosen. The boundary conditions at the sidewalls are more complex and a full consideration of them would require a separate and detailed analysis similar to that derived above for the interior. This situation is analogous to the case of rapid granular flows where special analyses of the boundary conditions to determine slip velocities and temperature jumps have been performed (cf. Hui *et al.* 1984; Gutt & Haff 1988; Jenkins & Richman 1986; Richman 1988; Richman & Chou 1988; Johnson *et al.* 1990). We shall not pursue such calculations in the present paper, but merely calculate $u - u_s$, the *difference* between the total velocity u and the wall slip velocity u_s, leaving the wall slip velocities undetermined. Accepting this deficiency, it turns out that we will be able to calculate all the other field variables necessary to describe the flow field.

As an example, let us consider the case in which the channel half-width b is large enough so that at the centreline the granular temperature T decays to zero. Hence, we have the condition $T(0) = 0$. As just noted, the gradients with respect to y vanish at the centreline; thus, $u'(0) = v'(0) = T'(0) = 0$. To perform the numerical solution for a specific example, we can stipulate the solids fraction at the centreline $v(0)$.

By applying (4.7) over the half-width b, we find that at the wall, $y = b$,

$$\sigma_{xy}\Big|_w = -\int_0^b \rho g \, \mathrm{d}y = -\overline{\rho} g b = \left[\mu \frac{\mathrm{d}u}{\mathrm{d}y}\right]_w = \sigma_{yy} \tan\delta, \tag{4.9}$$

where we have assumed that the wall shear stress can be expressed in terms of a wall friction angle $\delta \leqslant \arctan(\sin\phi)$. Since $\sigma_{yy} = -a$, this gives us a means to estimate a, i.e.

$$a = \frac{\overline{\rho} g b}{\tan\delta}, \tag{4.10}$$

and thence, to calculate the wall velocity gradient

$$\left[\frac{\mathrm{d}u}{\mathrm{d}y}\right]_w = \frac{-a\tan\delta}{\mu_w}. \tag{4.11}$$

By applying the expression (2.47) for $a(v, T)$ at the centreline where $T(0) = 0$, we can determine the constant a_0 that appears in (2.47). The granular temperature T_w at the wall $y = b$ can then be determined from (2.47).

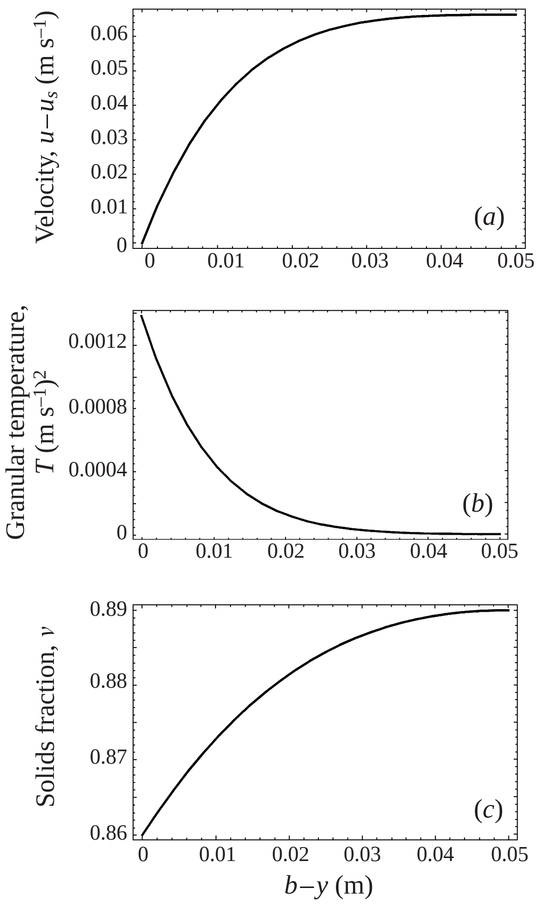

FIGURE 9. Variation of (*a*) velocity difference $u - u_s$, (*b*) granular temperature T, and (*c*) solids fraction ν, with distance from sidewall $b - y$. Channel half-width $b = 10d$, wall solids fraction $\nu_w = 0.86$.

While we do have the expression $a(\nu, T)$=const. fully specified, it is awkward to use to find a solution to the system of equations for u, ν, and T. It is more direct to simply numerically integrate the three ordinary differential equations: the x-momentum (4.4), y-momentum (4.5), and fluctuation energy (4.6) equations for u, ν, and T.

This integration is reasonably straightforward to perform using the `NDSolve` routine contained in the software system *Mathematica* 3.0 in a trial and error fashion. One integrates in the direction of $\eta = (b - y)$ from the wall ($\eta = 0$) to the centreline ($\eta = b$).

As described above, we can specify the boundary conditions for u, u', ν and T at the wall after making an initial estimate for the width-averaged density $\overline{\rho}$ in (4.10). Then, by trial and error, we choose T' at the wall such that $u' = 0$, $\nu' = 0$ and $T = 0$ at the centreline. Now, having the solids fraction (density) profile, we can integrate over the width to get a better estimate of $\overline{\rho}$, and repeat the whole sequence of steps until the solution has converged.

4.3. *Numerical results*

Several sets of computations were performed to predict the velocity, density and granular temperature profiles; some typical solutions are shown below in figures 9–12. All computations shown were carried out for a critical state internal friction

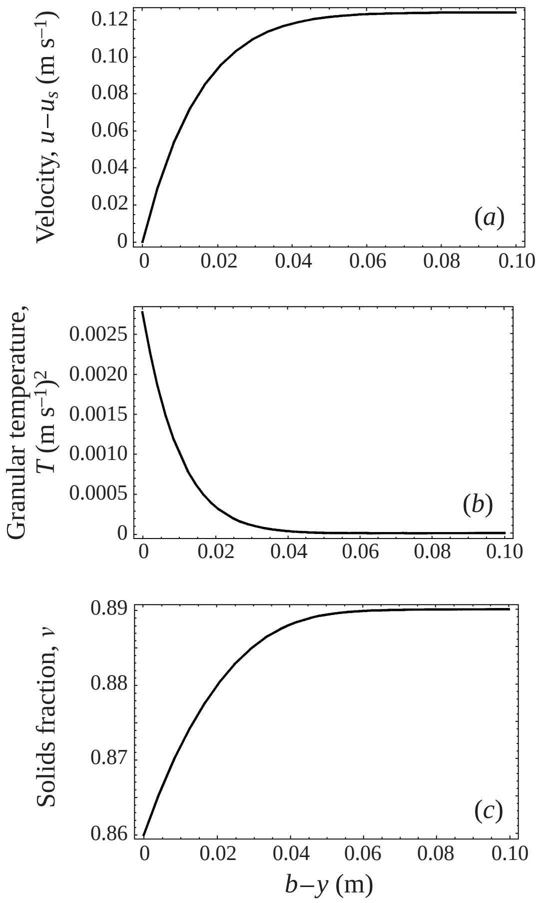

FIGURE 10. Same as figure 9 except for channel half-width $b = 20d$.

angle $\phi = 20°$, a wall friction angle $\delta = 18.88°$, mass density of the particles of $\rho_s = 1000$ kg m^{-3}, particle diameter $d = 5$ mm, minimum solids fraction $\nu_0 = 0.75$ and maximum solids fraction $\nu_\infty = 0.9$.

Figures 9–11 show predictions for a wall solids fraction $\nu_w = 0.86$, and a centreline solids fraction $\nu = 0.89$. Results for channel half-widths $b = 10d$, $20d$ and $40d$ are shown in figures 9, 10 and 11 respectively. As noted previously, we did not calculate the wall slip velocities u_s, and are only able to present the velocity difference $u - u_s$ versus distance from the sidewall $(b - y)$. Boundary layers of velocity, concentration and granular temperature develop next to the sidewalls. These boundary layers are roughly 10 particle diameters thick, with the concentration boundary layer being discernably larger than the velocity boundary layer. The thicknesses of the wall boundary layers remain approximately constant and the central plug flow region increases, as the overall channel width is increased. All of these results are similar to what was observed in the laboratory experiments of Pouliquen & Gutfraind (1996) and in the molecular dynamics computer simulations of Gutfraind & Pouliquen (1996). As the channel width increases, the present analysis predicts that both the velocity difference $u - u_s$, and the granular temperatures in the wall region increase.

Figure 12 shows predictions for a somewhat larger wall solids fraction $\nu_w = 0.87$, and a centreline solids fraction $\nu = 0.89$. The increase in the wall solids fraction yields

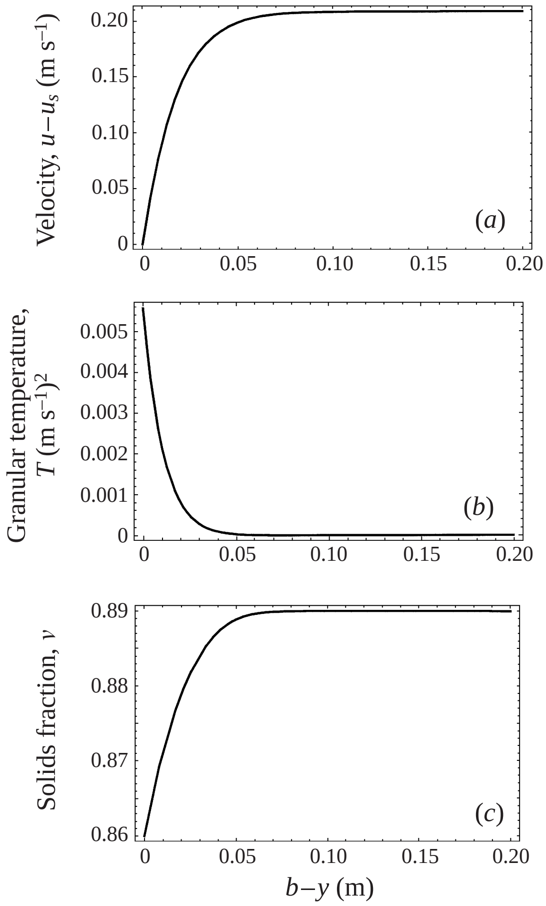

FIGURE 11. Same as figure 9 except for channel half-width $b = 40d$.

velocities and granular temperatures that are somewhat lower than the predictions shown in figure 11 for $\nu_w = 0.86$.

It should be mentioned that the computations have been performed by specifying the wall solids fraction ν_w and then calculating the velocities that could be associated with a total mass flow rate if we knew the wall slip velocity. In real physical experiments, it is likely that the flow rate would be fixed by some kind of 'control' at the outlet of the channel and the wall solids fraction would adjust itself so as to satisfy the equilibrium equations in the middle portion of the channel where there is no variation of flow quantitites in the streamwise direction.

5. Concluding remarks

We have described a first attempt to develop a simple model to handle slow flows of cohesionless, granular materials at relatively low stress levels. The basis of the model is the introduction of strain-rate fluctuations (Hibler 1977) into a critical state plasticity model that is similar in form to those proposed for quasi-static, pressure-dependent yielding of soils. The root-mean-square values of the strain-rate fluctuations were related to granular temperature. The forms of various expressions and constants

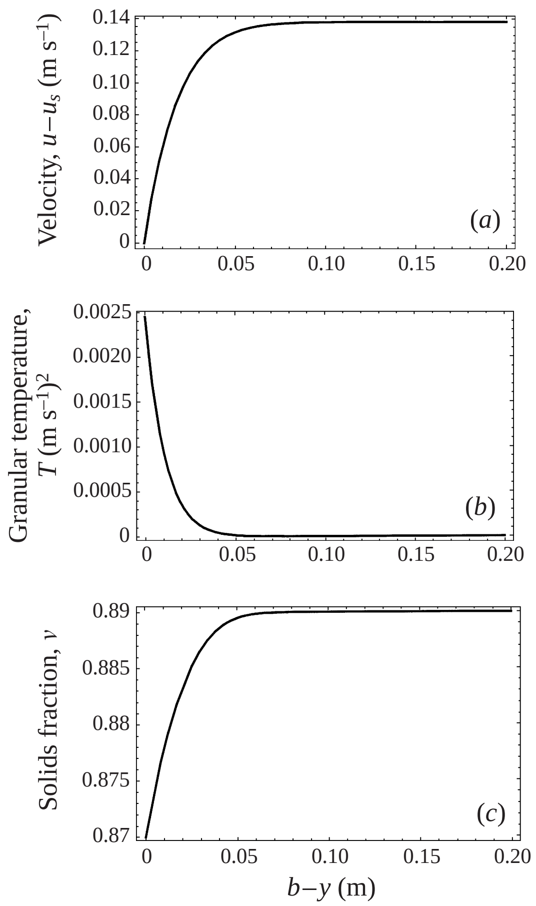

FIGURE 12. Same as figure 11 except for wall solids fraction $\nu_w = 0.87$.

involved in the model were chosen so that the predictions for high deformation rates are consistent with the collisional, granular flow kinetic theory of Jenkins (1987*a*).

At low deformation rates, the apparent form of the constitutive behaviour is similar to that of a liquid in the following sense. The effective viscosity decreases with increasing granular temperature as opposed to rapid granular flows in which viscosity increases with increasing granular temperature.

For the problem of simple shear flow, the resulting expressions for stresses were comprised of two parts: a rate-independent part, and a part that has a quadratic dependence on shear-rate. Similar Bingham like expressions have been proposed in the past on a more *ad hoc* basis.

A second problem of slow flow in a rough walled, vertical channel was also considered. The analysis was able to reproduce the observed experimental characteristics of nearly constant-thickness sidewall boundary layers (of about 10 particle diameters) with plug flow in the centre, regardless of the gap between the vertical sidewalls. The thickness of the solids fraction boundary layer was found to be larger than that of the velocity boundary layer. This same kind of behaviour is observed in both laboratory experiments and in molecular dynamics computer simulations.

The present analysis is merely a preliminary step to investigate the feasibility of the approach. Only two-dimensional particles and flows have been considered and the analysis should be extended to handle three-dimensional situations. We have solved the vertical chute flow problem without really considering the boundary conditions in detail. A more complete analysis of wall boundary conditions should be performed along the lines of what has been done in the case of kinetic theories of rapid granular flows. In particular, the wall slip velocities should be determined. Finally, it would be interesting to apply the present approach to consider other simple boundary value problems such as the free-surface flow in rough inclined channels, and free-surface flow down wedge-shaped piles of granular materials. This work is currently in progress.

Part of the work on this paper was carried out during a visit to the Institute for Theoretical Geophysics, Department of Applied Mathematics and Theoretical Physics, Cambridge University. I am grateful to Professors Herbert Huppert and David Crighton for their hospitality and encouragement during my stay there. My visit was made possible through the financial support of an NSERC International Collaborative Research Grant sponsored by the Royal Society, London and NSERC.

REFERENCES

ABAQUS/Standard 1994 Users Manual, Vols. I and II. Hibbitt, Karlsson & Sorensen, Inc., Pawtucket, Rhode Island.

Anderson, K. G., Jackson, R. 1992 A comparison of the solutions of some proposed equations of motion of granular materials for fully developed flow down inclines. *J. Fluid Mech.* **241**, 145–168.

Bagnold, R. A. 1954 Experiments on a gravity free dispersion of large solid spheres in a Newtonian fluid under shear. *Proc. R. Soc. Lond.* A **225**, 49–63.

Behringer, R. P. & Baxter, G. W. 1994 Pattern formation and complexity in granular flows. In *Granular Matter* (ed. A. Mehta), pp. 85–119. Springer.

Bolzon, G., Schrefler, B. A. & Zienkiewicz, O. C. 1996 Elastoplastic soil constitutive laws generalized to partially saturated states. *Géotechnique* **46**, 279–289.

Chapman, S. & Cowling, T. G. 1970 *The Mathematical Theory of Non-Uniform Gases*, 3rd edn. Cambridge University Press.

Chen, C. L. 1988 Generalized visco-plastic modelling of debris flow. *J. Hydraul. Engng ASCE* **114**, 237–258.

Chen, W. F. & Mizuno, E. 1990 *Nonlinear Analysis in Soil Mechanics*. Elsevier.

Clift, R. 1993 An Occamist review of fluidized bed modelling. In *Fluid–Particle Processes: Fundamentals and Applications* (ed. A. W. Weimer, J. C. Chen, L.-S. Fan & W. C. Yang). AIChE Symposium Series No. 296, vol. 89, pp. 1–17.

Cole, G. H. A. 1967 *The Statistical Theory of Classical Simple Dense Fluids*. Pergamon.

Cowin, S. C. 1977 The theory of static loads in bins. *Trans. ASME* E: *J. Appl. Mech.* **44**, 409–412.

Ding, J. & Gidaspow, D. 1990 A bubbling fluidization model using kinetic theory of granular flow. *AIChE J.* **36**, 523–538.

Drescher, A. & De Josselin De Jong, G. 1972 Photoelastic verification of a mechanical model for the flow of a granular material. *J. Mech. Phys. Solids* **20**, 337–351.

Eisenschitz, R. 1958 *Statistical Theory of Irreversible Processes*. Oxford University Press.

Fleck, N. A. & Hutchinson, J. W. 1997 Strain gradient plasticity. *Adv. Appl. Mech.* **33**, 295–361.

Gidaspow, D. 1994 *Multiphase Flow and Fluidization*. Academic.

Goldhirsch, I. 1995 Microstructures and kinetics in rapid granular flows. In *Traffic and Granular Flows*, pp. 251–266. World Scientific.

Goldshtein, A. & Shapiro, M. 1995 Mechanics of collisional motion of granular materials. Part 1. General hydrodynamic equations. *J. Fluid Mech.* **282**, 75–114.

Goldhirsch, I. & Sela, N. 1996 Origin of normal stress differences in rapid granular flows. *Phys. Rev.* E **54**, 4458–4461.

GOLDSHTEIN, A., SHAPIRO, M., MOLDAVSKY, L. & FICHMAN, M. 1995 Mechanics of collisional motion of granular materials. Part 2. Propagation through vibrofluidized granular layers. *J. Fluid Mech.* **287**, 349–382.

GUTFRAIND, R. & POULIQUEN, O. 1996 Study of the origin of shear zones in quasi-static vertical chute flows using discrete element simulations. *Mech. Mater.* **24**, 273–285.

GUTFRAIND, R., POULIQUEN, O. & SAVAGE, S. B. 1995 Molecular dynamic simulations of quasi-static granular chute flows. *Proc. 10th ASCE Engineering Mechanics Conference, Boulder, May* 21–24, vol. 2, pp. 794–797.

GUTT, G. M. & HAFF, P. K. 1988 Boundary conditions on continuum theories of granular flow. Brown Bag Preprint Series BB-70, Division of Physics, Mathematics, and Astronomy, Caltech, 24 pp.

HAFF, P. K. 1983 Grain flow as a fluid-mechanical phenomenon. *J. Fluid Mech.* **134**, 401–430.

HIBLER, W. D. III 1977 A viscous sea ice law as a stochastic average of plasticity. *J. Geophys. Res.* **82**, 3932–3938.

HIBLER, W. D. III 1979 A dynamic thermodynamic sea ice model. *J. Phys. Oceanogr.* **9**, 817–846.

HINCH, J. 1995 Introduction to mobile particulate systems. In *Mobile Particulate Systems* (eds. E. Guazzelli & L. Oger), pp. 1–21. Kluwer.

HOPKINS, M. A. & SHEN, H. H. 1992 A Monte-Carlo solution for rapidly shearing granular flows based on the kinetic theory of dense gases. *J. Fluid Mech.* **244**, 477–491.

HUI, K., & HAFF, P. K. 1986 Kinetic grain flow in a vertical channel. *Intl J. Multiphase Flow* **12**, 289–298.

HUI, K., HAFF, P. K., UNGAR, J. E. & JACKSON, R. 1984 Boundary conditions for high shear grain flows. *J. Fluid Mech.* **145**, 223–233.

JENIKE, A. W. 1961 Gravity flow of bulk solids. *Univ. Utah. Engng Exp. Stn Bull.* 108. University of Utah, Salt Lake City.

JENIKE, A. W. 1987 A theory of flow of particulate solids in converging and diverging channels based on a conical yield function. *Powder Technol.* **50**, 229–236.

JENKINS, J. T. 1987*a* Balance laws and constitutive relations for rapid flows of granular materials. In *Constitutive Models of Deformations* (ed. J. Chandra & R. Srivastav), pp. 109–119. SIAM.

JENKINS, J. T. 1987*b* Rapid flows of granular materials. In *Non-classical Continuum Mechanics: Abstract Techniques and Applications* (ed. R. J. Kops & A. A. Lacey), pp. 213–224. Cambridge University Press,

JENKINS, J. T. & MCTIGUE, D. F. 1990 Transport processes in concentrated suspensions: the role of particle fluctuations. In *Two Phase Flows and Waves* (ed. D. D. Joseph and D. G. Schaeffer), pp. 70–79. Springer.

JENKINS, J. T. & MCTIGUE, D. F. 1995 Viscous fluctuations and the rheology of concentrated suspensions. submitted to *J. Fluid Mech.*

JENKINS, J. T. & RICHMAN, M. W. 1985 Grad's 13-moment system for a dense gas of inelastic spheres. *Arch. Rat. Mech. Anal.* **87**, 355–377.

JENKINS, J. T. & RICHMAN, M. W. 1986 Boundary conditions for plane flows of smooth nearly elastic, circular disks. *J. Fluid Mech.* **171**, 53–69.

JENKINS, J. T. & RICHMAN, M. W. 1988 Plane simple shear of smooth inelastic circular disks: the anisotropy of the second moment in the dilute and dense limits. *J Fluid Mech.* **192**, 313–328.

JENKINS, J. T. & SAVAGE, S. B. 1983 A theory for the rapid flow of identical, smooth, nearly elastic particles. *J. Fluid Mech.* **130**, 186–202.

JOHNSON, P. C. & JACKSON, R. 1987 Frictional-collisional constitutive relations for granular materials, with application to plane shearing. *J. Fluid Mech.* **176**, 67–93.

JOHNSON, P. C., NOTT, P. & JACKSON, R. 1990 Frictional-collisional equations of motion for particulate flows and their application to chutes. *J. Fluid Mech.* **210**, 501–535.

KRAUSZ, A. S. & EYRING, H. 1975 *Deformation Kinetics*. Wiley.

KRUYT, N. 1990 Density effects in two–dimensional bins. *Trans. ASME* E: *J. Appl. Mech.* **57**, 1032–1035.

LADE, P. V. 1977 Elasto–plastic stress–strain theory for cohesionless soil with curved yield surfaces. *Intl J. Solids Struct.* **13**, 1019–1035.

LADE, P. V. & PRABUCKI, M.-J. 1995 Softening and preshearing effects in sand. *Soils and Found., Japanese Geotechn. Soc.* **35**(4), 93–104.

Lade, P. V. & Pradel, D. 1990 Instability and plastic flow of soils. I: Experimental observations. *J. Engng Mech., ASCE* **116**, 2532–2550.

Lan, Y. & Rosato, A. D. 1995 Macroscopic behavior of vibrating beds of smooth inelastic spheres. *Phys. Fluids* **7**, 1818–1831.

Louge, M., Mastorakos, E. & Jenkins, J. T. 1991 The role of particle collisions in pneumatic transport. *J. Fluid Mech.* **231**, 345–359.

Lun, C. K. K. 1991 Kinetic theory for granular flow of dense, slightly inelastic slightly rough spheres. *J. Fluid Mech.* **233**, 539–559.

Lun, C. K. K. & Bent, A. A. 1994 Numerical simulations of inelastic frictional spheres in simple shear flow. *J. Fluid Mech.* **258**, 335–353.

Lun, C. K. K. & Savage, S. B. 1987 A simple kinetic theory for granular flow of rough, inelastic spherical particles. *Trans. ASME* E: *J. Appl. Mech.* **54**, 47–53.

Lun, C. K. K., Savage, S. B., Jeffrey, D. J. & Chepurniy, N. 1984 Kinetic theories for granular flow: inelastic particles in Couette flow and slightly inelastic particles in a general flowfield. *J. Fluid Mech.* **140**, 223–256.

McQuarrie, D. A. 1976 *Statistical Mechanics.* Harper and Row.

McTigue, D. F. & Jenkins, J. T. 1992 Channel flow of a concentrated suspension. In *Advances in Micromechanics of Materials* (ed. H. H. Shen, M. Satake, M. Mehrabadi, C. S. Chang & C. S. Campbell), pp. 381–390. Elsevier.

Mroz, Z., Norris, V. A. & Zienkiewicz, O. C. 1979 Application of an anisotropic hardening model in the analysis of elasto-plastic deformation of soils. *Géotechnique* **29**, 1–34.

Mühlhaus, H.-B. & Vardoulakis, I. 1987 The thickness of shear bands in granular materials. *Géotechnique* **37**, 271–283.

Natarajan, V., Hunt M. & Taylor, E. 1995 Local measurements of velocity fluctuations and diffusion coefficients for granular material flow. *J. Fluid Mech.* **304**, 1–25.

Nedderman, R. M. & Laohakul, C. 1980 The thickness of the shear zone of flowing granular materials. *Powder Technol.* **25**, 91–100.

Oda, M., Nemat-Nasser, S. & Konishi, J. 1985 Stress–induced anisotropy in granular masses. *Soils and Foundations* **25**, 85–97.

Pouliquen, O. & Gutfraind, R. 1996 Stress fluctuations and shear zones in quasi-static granular chute flows. *Phys. Rev.* E **53**, 552–561.

Pradel, D. & Lade, P. V. 1990 Instability and plastic flow of soils. II: Analytical Investigation. *J. Engng Mech., ASCE.* **116**, 2551–2566.

Richman, M. W. 1988 Boundary conditions based upon a modified Maxwellian velocity distribution for flows of identical, smooth, nearly elastic, spheres. *Acta Mechanica* **75**, 227–240.

Richman, M. W. 1989 The source of second moment in dilute granular flows of highly inelastic spheres. *J. Rheol.* **33**, 1293–1306.

Richman, M. W. & Chou, C. S. 1988 Boundary effects on granular shear flows of smooth disks. *Math. Phys. Z. Angew.* **39**, 885–901.

Richman, M. W. & Chou, C. S. 1992 Constitutive theory for homogeneous granular shear flows of highly inelastic spheres. submitted to *J. Rheol.*

Savage, S. B. 1979 Gravity flow of cohesionless granular materials in chutes and channels. *J. Fluid Mech.* **92**, 53–96.

Savage, S. B. 1983 Granular flows down rough inclines - review and extension. In *Mechanics of Granular Materials: New Models and Constitutive Relations* (ed. J. T. Jenkins & M. Satake), pp. 261–82. Elsevier.

Savage, S. B. 1984 The mechanics of rapid granular flows. *Adv. Appl. Mech.* **24**, 289–366.

Savage, S. B. 1992 Numerical simulation of Couette flow of granular materials; Spatio–temporal coherence and $1/f$ noise. In *Physics of Granular Media* (ed. D. Bideau & J. Dodds), pp. 343–362. Nova Scientific Publishers.

Savage, S. B. 1995 Flow of granular materials. In *Proc. 15th Canadian Congress of Applied Mechanics, CANCAM 95* (ed. B. Tabarrok & S. Dost), pp. 62–73. University of Victoria.

Savage, S. B. & Jeffrey, D. J. 1981 The stress tensor in a granular flow at high shear rates. *J. Fluid Mech.* **110**, 255–272.

Schofield, A. N. & Wroth, C. P. 1968 *Critical State Soil Mechanics.* McGraw-Hill.

Sela, N. & Goldhirsch, I. 1998 Hydrodynamical equations for rapid flows of smooth inelastic spheres Burnett order. *J. Fluid Mech.* **361**, 41–74.

SELA, N., GOLDHIRSCH, I. & NOSKOWICZ, S. H. 1996 Kinetic theoretical study of a two-dimensional granular gas to Burnett order. *Phys. Fluids* **8**, 2337–2353.

SINCLAIR, J. L. & JACKSON, R. 1989 Gas-particle flow in a vertical pipe with particle–particle interactions. *AIChE J.* **35**, 1473–1486.

SUNDARAM, V. & COWIN, S. C. 1979 A reassessment of static bin pressure experiments. *Powder Technol.* **22**, 23–32.

TAKAHASHI, T. 1981 Debris flow. *Ann. Rev. Fluid. Mech.* **13**, 57–77.

TAKAHASHI, T. 1991 *Debris Flow*. A. A. Balkema.

TAKAHASHI, H. & YANI, H. 1973 Flow profile and void fraction of granular solids in a moving bed. *Powder Technol.* **7**, 205–214.

TSUBAKI, T. & HASHIMOTO, H. 1983 Interparticle stresses and characteristics of debris flows. *J. Hydrosc. Hydr. Engng*, **1**, 67–82.

VARDOULAKIS, I. G. & SULEM, J. 1995 *Bifurcation Analysis in Geomechanics*. Chapman & Hall.

VERMEER, P. A. 1984 A five-constant model unifying well-established concepts. In *Constitutive Relations for Soils* (ed. G. Gudehus, F. Darve, & I. Vardoulakis), pp. 175–197. A. A. Balkema.

WALTON, O. R. & BRAUN, R. L. 1986*a* Stress calculations for assemblies of inelastic spheres in uniform shear. *Acta Mechanica* **63**, 73–86.

WALTON, O. R. & BRAUN, R. L. 1986*b* Viscosity and temperature calculations for assemblies of inelastic frictional disks. *J. Rheol.* **30**, 949–980.

WALTON, O. R., BRAUN, R. L., MALLON, R. G., & CERVELLI, D. M. 1987 Particle-dynamics calculations of gravity flow of inelastic, frictional spheres. In *Micromechanics of Granular Materials* (ed. M. Satake & J. T. Jenkins), pp. 153–162. Elsevier.

WOOD, D. M. 1990 *Soil Behaviour and Critical State Soil Mechanics*. Cambridge University Press.

WOODS, L. C. 1975 *The Thermodynamics of Fluid Systems*. Oxford University Press.

WOODS, L. C. 1993 *A Introduction to the Kinetic Theory of Gases and Magnetoplasmas*. Oxford University Press.